

CLASSIFICATION OF 2-DIMENSIONAL GRADED NORMAL HYPERSURFACES WITH $a(R) \leq 6$.

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INTRODUCTION

Isolated weighted homogeneous surface singularities of type $R = k[X, Y, Z]/(f)$ are extensively studied by V.I. Arnold, H. Pinkham [P1] and K. Saito and many other authors. Especially, K. Saito [S1] studies these in terms of “regular system of weights”. On the other hand, from the view point of commutative ring theory, the a -invariant $a(R)$ defined in [GW] (in Saito’s paper, the notation $\varepsilon = -a(R)$ is used) and also, such singularities can be constructed by so called DPD (Dolgachev-Pinkham-Demazure) construction [Dol][P2][Dem].

The author tried the classification of all the possible weights for given $a(R)$ using DPD construction of normal graded rings and it turned out that the procedure is so simple and nearly automatic.

Although this classification is “known” in the literature (cf. [S1], [Wag]), it seems that the algebraic or ring-theoretic classification is not done yet.

Another good point of our classification is that we can draw the “graph” of the resolution of the singularity of $\text{Spec}(R)$ instantly from the expression as $R = R(X, D)$.

Below, we present the classification of such singularities with $0 < a(R) \leq 6$. We prove in general that for a given $\alpha > 0$, the number of types of R for 2-dimensional normal graded hypersurfaces with $a(R) = \alpha$ is finite.

1. PRELIMINARIES

Let $R = \oplus_{n \geq 0} R_n = k[u, v, w]/(f)$ be a 2-dimensional graded normal hypersurface, where k is an algebraically closed field of any characteristic. We always assume that the grading of R is given so that $R_n \neq 0$ for $n \gg 0$. We put $X = \text{Proj}(R)$. Since $\dim R = 2$ and R is normal, X is a smooth curve. Then by the construction of Dolgachev, Pinkham and Demazure ([Dem], [P2]), there is an ample \mathbb{Q} -Cartier divisor D (that is, ND is an ample divisor on X for some positive integer N), such that

$$R \cong R(X, D) = \oplus_{n \geq 0} H^0(X, O_X(nD)) \cdot T^n \subset k(X)[T]$$

as graded rings, where T is a variable over $k(X)$ and

$$H^0(X, O_X(nD)) = \{f \in k(X) \mid \text{div}_X(f) + nD \geq 0\} \cup \{0\}.$$

We denote by $[nD]$ the integral part of nD so that $H^0(X, O_X(nD)) = H^0(X, O_X([nD]))$.

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Now, let us begin the classification. In the following, X is a smooth curve of genus g and D is a fractional divisor on X such that ND is an ample integral (Cartier) divisor for some $N > 0$.

We always denote

$$(1.0.1) \quad D = E + \sum_{i=1}^r \frac{p_i}{q_i} P_i \quad (p_i, q_i \text{ are positive integers and } \forall i, (p_i, q_i) = 1),$$

where E is an integral divisor. In this case, we denote

$$(1.0.2) \quad \text{frac}(D) = \sum_{i=1}^r \frac{q_i - 1}{q_i} P_i.$$

At the same time, by our assumption $R \cong k[u, v, w]/(f)$. If $\deg(u, v, w; f) = (a, b, c; h)$, then by [GW],

$$(1.0.3) \quad a(R) = h - (a + b + c).$$

We always assume $\deg(u, v, w; f) = (a, b, c; h)$ and also that $a \leq b \leq c$. We call $(a, b, c; h)$ the *type* of R . In this paper, we will determine all the possible types of R for $0 < a(R) \leq 6$ ¹.

We list our tools to classify.

Proposition 1.1. (Fundamental formulas) *Assume that $R = R(X, D) \cong k[u, v, w]/(f)$ with $\deg(u, v, w; f) = (a, b, c; h)$ and $a(R) = h - (a + b + c) = \alpha$. Then we have the following equalities.*

(1) [W] *Since R is Gorenstein with $a(R) = \alpha$, we have*

$$\alpha D \sim K_X + \text{frac}(D) = K_X + \sum_{i=1}^r \frac{q_i - 1}{q_i} P_i,$$

where, in general, $D_1 \sim D_2$ means that $D_1 - D_2 = \text{div}_X(\phi)$ for some $\phi \in k(X)$.

In particular, the genus g of X is given by

$$g = g(X) = H^0(X, \mathcal{O}_X(K_X)) = \dim R_\alpha.$$

(2) [To] *If $P(R, t) = \sum_{n \geq 0} \dim R_n t^n$ is the Poincare series of R , then*

$$\lim_{t \rightarrow 1} (1 - t)^2 P(R, t) = \deg D.$$

Since $P(R, t) = \frac{1 - t^h}{(1 - t^a)(1 - t^b)(1 - t^c)}$ in our case, we have

$$\deg D = \frac{h}{abc} = \frac{\alpha}{abc} + \frac{1}{ab} + \frac{1}{ac} + \frac{1}{bc}.$$

Note that the latter expression is a decreasing function of a, b, c .

Lemma 1.2. [OW] *Let $R = k[x, y, z]/(f)$ be a normal graded ring with type $(a, b, c; h)$ with $h = a + b + c + \alpha$. Then*

(1) *If h is not a multiple of c , then either $h - a$ or $h - b$ is a multiple of c . The same is true for a, b . Namely, at least one of $h, h - a, h - c$ is a multiple of b and at least one of $h, h - b, h - c$ is a multiple of a .*

¹See Remark 2.5 for the case $\alpha < 0$

(2) If a prime number p divides two of a, b, c , then p divides h . In particular, if α is even, then at most one among a, b, c is even.

(3) $c \leq a + b + \alpha$.

Proof. If $R = k[x, y, z]/(f)$ is normal, then f must contain monomial of the form x^n or x^ny or x^nz . The statement (1), (3) follows from this fact. As for (2), It is easy to see that if p divides a, b and not h , then f must be divisible by z . \square

We list some properties of R when R is Gorenstein.

Lemma 1.3. *Let $R = R(X, D)$ be a normal graded ring with D as in (1.0.1) and assume that R is Gorenstein with $a(R) = \alpha$. Then we have the following formulas.*

- (1) *For every n , $0 \leq n \leq \alpha$, we have $\deg[nD] + \deg[(\alpha - n)D] = \deg[\alpha D] = 2g - 2$, where g is the genus of X .*
- (2) $[(\alpha + 1)D] = K_X + E + \sum_{i=1}^r P_i$.
- (3) $[(2\alpha + 1)D] = 2K_X + E + 2\sum_{p_i \geq 2} P_i + \sum_{p_i=1} P_i$.

Proof. This follows easily from $\alpha D = K_X + \text{frac}(D)$. \square

Next we recall some fundamental property of $p_g(R)$.

Definition 1.4. If $X \rightarrow \text{Spec}(R)$ is a resolution of singularities of R , then the geometric genus of R , $p_g(R)$ is defined by

$$p_g(R) = \dim_k H^1(X, \mathcal{O}_X).$$

When R is a Gorenstein graded ring, it is proved in [W] that

$$(1.5.1) \quad p_g(R) = \sum_{n=0}^{a(R)} \dim_k R_n.$$

In the following, we denote $a(R) = \alpha$ to avoid confusion with a (the minimal positive degree with $R_a \neq 0$).

Remark 1.5. [Dem] If $R = R(X, D)$ with $D = E + \sum_{i=1}^r \frac{p_i}{q_i} P_i$, the “graph” of the resolution of singularity of $\text{Spec}(R)$ is a so called “star-shaped” graph with “central curve” X with self intersection $X^2 = -[D]$ and branch of \mathbb{P}^1 ’s intersecting at $P_i \in X$ with self intersection number $-b_1, \dots, -b_s$ if the continued-fraction expression of $\frac{q_i}{q_i - p_i}$ is as $\frac{q_i}{q_i - p_i} = b_1 - \frac{1}{b_2 - \frac{1}{b_3 - \dots}}$

2. GENERAL RESULTS FOR GIVEN $\alpha = a(R) > 0$.

First, we will show the finiteness of the types $(a, b, c; h)$ for given $\alpha = a(R) > 0$.

Theorem 2.1. *If we fix $\alpha = a(R) > 0$, the number of $(a, b, c; h)$ for a normal graded ring $R = k[x, y, z]/(f)$ with $h = a + b + c + \alpha$ is finite².*

²See Remark 2.5 for the case $\alpha \leq 0$.

In the rest of this section, we fix $\alpha > 0$, always assume that $a \leq b \leq c, h = a + b + c + \alpha$ and $(a, b, c) = 1$. The proof is done in a series of lemmas. Note that by Lemma 1.2, it suffices to show that the number of possible (a, b) is finite. Also, for simplicity, we assume that $c \geq \alpha$.

Lemma 2.2. (1) We have $\deg D \leq \frac{4}{ab}$.

(2) If $g \geq 2$, then $ab < \frac{4\alpha}{2g-2}$.

Proof. (1) By our assumption $c \geq \alpha$, $\deg D = \frac{a+b+c+\alpha}{abc} \leq \frac{4}{ab}$.

(2) Since $\alpha D \sim K_X + \text{frac}(D)$, $\deg D \geq \frac{2g-2}{\alpha}$ and the LHS is less than $\frac{4}{ab}$. \square

Lemma 2.3. If $g = 1$, then $ab < 8\alpha$.

Proof. As in the previous lemma, since $\alpha \deg D \sim \text{frac}(D)$ and $\deg \text{frac}(D) \geq \frac{1}{2}$, we have $\deg D = \frac{a+b+c+\alpha}{abc} \geq \frac{1}{2\alpha}$. Hence we have $\frac{4}{ab} > \frac{1}{2\alpha}$. \square

Lemma 2.4. If $g = 0$, then $ab < 168\alpha$.

Proof. We have $\alpha \deg D = \text{frac}(D) - 2$ and it is easy to show that if $\sum_{i=1}^r \frac{q_i - 1}{q_i} - 2 > 0$, then the minimal such value of the LHS is $\frac{1}{42}$. \square

By these Lemmas, we have proved that the number of types of R is finite for a fixed $\alpha = a(R)$.

Remark 2.5. Although the following results are somewhat “known” in the literature, I include the cases of $\alpha = a(R) \leq 0$ for the completeness.

If $\alpha = 0$, then $K_X = 0$ and $\text{frac}(D) = 0$. Hence $g(X) = 1$ and $\deg D \leq 3$. We have $(a, b, c; h) = (1, 2, 3; 6), (1, 2, 2; 4), (1, 1, 1; 3)$ according to $\deg D = 1, 2, 3$ respectively.

If $\alpha < 0$, then since $\deg(K_X + \text{frac}(D)) < 0$, we have $X \cong \mathbb{P}^1, r \leq 3$ and if $r = 3$, (q_1, q_2, q_3) is either $(2, 2, n), (2, 3, 3), (2, 3, 4)$ or $(2, 3, 5)$. If we put $D = -(K_X + \text{frac}(D))$ in these cases, then we have $\alpha = -1$ and we get $(a, b, c; h) = (2, n, n+1; 2n+2), (3, 4, 6; 12), (4, 6, 9; 18), (6, 10, 15)$ respectively, corresponding to $(D_{n+2}), (E_6), (E_7), (E_8)$ singularities. Note that the number of types is infinite in this case.

Since in the case $\alpha < 0$ and $r = 3$, there is no integer $n < -1$ with $nD \sim K_X + \text{frac}(D)$, if $\alpha < -1$, then $r \leq 2$ and putting $P_1 = (0)$ and $P_2 = (\infty)$, then R is generated by monomials and is isomorphic to a ring of the form $k[u, v, w]/(uv - w^n)$. Note that in this case, the type $(a, b, c; h)$ is not uniquely determined by the ring. On the contrary, if $\alpha \geq 0$ or $r \geq 3$, the type $(a, b, c; h)$ is uniquely determined by the ring $k[X, Y, Z]/(f)$ since the resolution of $\text{Spec}(R)$ given in 1.5 is a minimal good resolution³ and conversely a minimal good resolution determines R as a graded ring. Related general statement can be found in [S0].

³A resolution whose exceptional set consists of smooth curves with normal crossings and which is minimal resolution satisfying this condition; namely, which contains no (-1) curve intersecting to at most 2 other irreducible curves.

In the following, we consider only the case $\alpha > 0$.

Question 2.6. Assume that $R = k[X, Y, Z, W]/(f)$ is graded with type $(a, b, c, d; h)$ and has isolated singularity. For a given $\alpha > 0$, is the number of $(a, b, c, d; h)$ with $\alpha = h - (a + b + c + d)$ finite ?

The following Theorem is the main result of [S2]. We give a proof here because it is much simpler by our method than the one given there.

Theorem 2.7. *For any given $\alpha > 0$, we have either $R_{\alpha-1} \neq 0$ or $R_{\alpha+1} \neq 0$.*

Proof. Let $R = R(X, D)$ and g be the genus of X and put $D = E + \sum_{i=1}^r \frac{p_i}{q_i} P_i$ as in (1.0.1). By Lemma 1.3 (2), we have always

$$[(\alpha + 1)D] = K_X + E + \sum_{i=1}^r P_i.$$

Since $\deg D > 0$, we have $\deg E \geq 1 - r$ and $\deg[(\alpha + 1)D] \geq 2g - 1$. Hence if $g > 0$, we have always $R_{\alpha+1} \neq 0$.

If $g = 0$, we have

$$\deg[(\alpha - 1)D] = -2 - \deg E \quad \text{and} \quad \deg[(\alpha + 1)D] = -2 + \deg E + r.$$

Hence if $\deg E \leq -2$, we have $R_{\alpha-1} \neq 0$ and if $\deg E \geq -1$, then $R_{\alpha+1} \neq 0$ since $r \geq 3$. \square

Next we will show that if $p_g(R) = 1$, then $\alpha \leq 7$ and that α is bounded if $p_g(R)$ is bounded. These result also appears in Saito's paper [S1], but we will show it by our method. We can easily distinguish R with $p_g(R) = 1$ and $\alpha \leq 6$ from our table.

Theorem 2.8. (1) *If $p_g(R) = r$ is fixed, then α is bounded.*

(2) *If $p_g(R) = 1$, then $\alpha \leq 7$. If $p_g(R) = 1$ and $\alpha = 7$, then the type of R is either $(8, 10, 15; 40)$, $(8, 10, 25; 50)$ or $(8, 9, 12; 36)$.*

Proof. (1) If $p_g(R) = r$, then $a \leq \alpha/r$ by formula (1.5.1). Hence if α tends to ∞ , then $\deg D \leq c\alpha^{-2}$ for some constant c by Proposition 1.1. While we have shown in Lemma 2.2 to 2.4 that $\deg D \geq c'\alpha^{-1}$ for some constant c' .

(2) Put $R = R(X, D)$ and g be the genus of X . If $g > 0$, then we have $\dim R_\alpha = g$ and hence we have always $p_g(R) \geq g + 1 \geq 2$.

Now we assume $g = 0$ and $p_g(R) = 1$ and as always, we write $D = E + \sum_{i=1}^r \frac{p_i}{q_i} P_i$.

By formula (1.5.1), we have $R_n = 0$ for $1 \leq n \leq \alpha$. Hence by Theorem 2.7, $a = \alpha + 1$. Since $R_n \neq 0$ if and only if $\deg[nD] \geq 0$, and

$$(2.8.1) \quad [nD] + [(\alpha - n)D] = [\alpha D] = -2 \quad \text{for } 1 \leq n \leq \alpha - 1,$$

we have

$$(2.8.2) \quad \deg[nD] = -1 \quad \text{for } 1 \leq n \leq \alpha - 1.$$

In particular, if $\alpha \geq 2$ and $p_g(R) = 1$, then $\deg E = -1$.

Also, since $[(\alpha + 1)D] = -2 + \deg E + r$ and $\dim R_{\alpha+1} \leq 2$, we get $r \leq 4$. If $r = 4$, then $\alpha \leq 2$ by the following Lemma 2.9. So, we assume $r = 3$ in the following and assume $q_1 \leq q_2 \leq q_3$.

Since $a = \alpha + 1$ in this case, every element of degree $\leq 2\alpha + 1$ is a member of minimal generating set of R . Hence we must have

$$(2.8.3) \quad \sum_{n=\alpha+1}^{2\alpha+1} \dim R_n \leq 3.$$

On the other hand, since $\deg[nD] = -1$ for $1 \leq n \leq \alpha - 1$ and $(\alpha + n)D \sim K_X + \text{frac}(D) + nD$ for every n , $1 \leq n \leq \alpha - 1$, we have

$$(2.8.4) \quad R_{\alpha+n} \neq 0 \iff \text{no } q_i \text{ divides } n.$$

Thus, we conclude that every n , $1 \leq n \leq \alpha - 1$, except at most 3 is a multiple of some q_i . This excludes the possibility $\alpha = 6$ since, every q_i should be relatively prime to 6 and $n = 1, 2, 3, 4$ are not multiple of any q_i .

Next, we assume that α is even and ≥ 8 . Then since every q_i should be odd, we should have $(a, b, c) = (\alpha + 1, \alpha + 2, \alpha + 4)$ by (2.8.4). But this contradicts the condition of 1.2 (2) and hence this case does not occur.

Now, we assume that α is odd and $\alpha \geq 7$. To proceed further, we note that if $q_1 = 2$ and $q_2 = 3$, then $R_6 \neq 0$. Hence we have either $R_{\alpha+2} \neq 0$ or $R_{\alpha+3} \neq 0$.

Thus, we have either

$$(2.8.5) \quad (a, b, c; h) = (\alpha + 1, \alpha + 2, c; c + 3\alpha + 3) \quad \text{or}$$

$$(2.8.6) \quad (a, b, c; h) = (\alpha + 1, \alpha + 3, c; c + 3\alpha + 4)$$

Now we check the condition of 1.2 (1). In case (2.8.5), c should divide either $3\alpha + 3$, $2\alpha + 2$ or $2\alpha + 1$. Then we should have $(a, b, c; h) = (\alpha + 1, \alpha + 2, 2\alpha + 1; 5\alpha + 4)$ or $(\alpha + 1, \alpha + 2, (3\alpha + 3)/2; 9(\alpha + 1)/2 + 1)$. In the former case, $\alpha + 2$ must divide either $5\alpha + 4$, $3\alpha + 3$ or $4\alpha + 3$ and we can see such cases do not occur. In the latter case, $\alpha + 2$ should divide $9(\alpha + 1)/2$, the only solution is $\alpha = 7$, then $(a, b, c; h) = (8, 9, c; c + 24)$ and only possibility is and

$$(a, b, c; h) = (8, 9, 12; 36). \quad D = \frac{2}{3}P_1 + \frac{1}{4}P_2 + \frac{1}{8}P_3 - Q.$$

In case (2.8.6), c should divide either $3\alpha + 4$, $2\alpha + 1$ or $2\alpha + 3$. Then we should have $(a, b, c; h) = (\alpha + 1, \alpha + 3, 2\alpha + 1; 5\alpha + 5)$, $(\alpha + 1, \alpha + 3, 2\alpha + 3; 5\alpha + 7)$ or $(\alpha + 1, \alpha + 3, 3\alpha + 4; 6\alpha + 8)$.

As in the previous case, since we have assumed $\alpha \geq 7$, the only solutions are

$$(8, 10, 15; 40) \quad D = \frac{1}{2}P_1 + \frac{2}{5}P_2 + \frac{2}{15}P_3 - Q,$$

$$(8, 10, 25; 50) \quad D = \frac{1}{2}P_1 + \frac{2}{5}P_2 + \frac{1}{8}P_3 - Q.$$

□

Lemma 2.9. *Let $R = R(X, D)$ be as in Theorem 2.8, with $g = 0$ and $r = 4$. Then if $p_g(R) \leq 1$, then $\alpha \leq 2$.*

Proof. If $\alpha \geq 3$, then by formula (2.8.2), $\deg E = -1$ and $\deg[2D] = -1$. Hence the number of i with $q_i = 2$ is at most 1 and $\deg \alpha D \geq \frac{1}{2} + 3 \cdot \frac{2}{3} = \frac{5}{2}$. On the other

hand, since the type of R is $(\alpha + 1, \alpha + 1, \alpha + 2 : 4\alpha + 4)$, $\deg D = \frac{4}{(\alpha + 1)(\alpha + 2)}$, which is smaller than $\frac{5}{4\alpha}$. A contradiction ! \square

3. THE CLASSIFICATION OF THE HYPERSURFACES WITH $a(R) \leq 6$.

Now let us begin the classification of normal graded surface $R = k[x, y, z]/(f)$ with $a(R) = \alpha \leq 6$.

The case $\alpha = 1$

Henceforce, we put $D = K_X + \sum_{i=1}^r \frac{q_i - 1}{q_i} P_i$. From 1.1 (3), the maximal value of $\deg D$ is taken when $a = b = c = 1$ and $\deg D = 4$ in that case.

Case 1 - A. The case $g > 0$.

Assume that $g \geq 1$. Since $\deg(D) \leq 4$, and $\deg D \geq \deg K_X = 2g - 2$, $g \leq 3$ and if $g = 3$, $D = K_X$. We list the cases by giving the form of D and $(a, b, c; h)$. We can easily deduce the general form of the equation f from this data. Also, if f with the given weight has an isolated singularity, then $k[u, v, w]/(f) \cong R(X, D)$, where D is a divisor of given form.

$$(1-A-1) \quad g = 3, D = K_X; \quad (1, 1, 1; 4).$$

Next, consider the case $g = 2$. Note that $\dim R_1 = \dim H^0(K_X) = g = 2$, we have $a = b = 1$ and $\deg D = 1 + \frac{3}{c} \leq \frac{5}{2}$ ($c \geq 2$). Since, either $\deg D = 2$, $D = K_X$ or $\deg D \geq \frac{5}{2}$, we have 2 cases.

$$(1-A-2) \quad g = 2, D = K_X; \quad (1, 1, 3; 6).$$

$$(1-A-3) \quad g = 2, D = K_X + \frac{1}{2}P; \quad (1, 1, 2; 5).$$

Next, assume $g = 1$. In this case, $a = 1$, $2 \leq b \leq c$ and the maximal value of $\deg D$ is $\frac{3}{2}$. Since on the other hand, $\deg D \geq \frac{r}{2}$ and thus $r \leq 3$ and if $r = 3$, $D = \frac{1}{2}(P_1 + P_2 + P_3)$.

$$(1-A-4) \quad g = 1, D = \frac{1}{2}(P_1 + P_2 + P_3); \quad (1, 2, 2; 6).$$

Also, since $\dim R_2 = r$, if $r = 2$, then $a = 1, b = 2, c \geq 3$, $\deg(D) \leq \frac{7}{6}$.

$$(1-A-5) \quad g = 1, D = \frac{1}{2}(P_1 + P_2); \quad (1, 2, 4; 8).$$

$$(1-A-6) \quad g = 1, D = \frac{1}{2}P_1 + \frac{2}{3}P_2; \quad (1, 2, 3; 7).$$

If $g = 1$ and $D = \frac{q-1}{q}P$, we have $q-1$ new generators in degrees $1, 3, \dots, q$. Hence $q \leq 4$.

$$(1-A-7) \quad g = 1, D = \frac{1}{2}P; \quad (1, 4, 6; 12).$$

$$(1-A-8) \quad g = 1, D = \frac{2}{3}P; \quad (1, 3, 5; 10).$$

$$(1-A-9) \quad g = 1, D = \frac{3}{4}P; \quad (1, 3, 4; 9).$$

We have 9 types when $g \geq 1$.

Case 1- B. The case $g = 0$ and $r \geq 4$.

Since $\deg(K_X) = -2$ and $\deg D > 0$, we have $r \geq 3$. On the other hand, since $R_1 = H^0(K_X) = 0$, $a \geq 2, c \geq 3$, we have $\deg D \leq \frac{2}{3} < 1$. Since $\deg D \geq -2 + r/2$, we have $r \leq 5$.

Now, since $\deg[2D] = r - 4$, $\dim R_2 = 2, 1$, respectively, if $r = 5, 4$.

Thus if $r = 5$, then $a = b = 2$ and $c \geq 3$. Since $3 \cdot \frac{1}{2} + 2 \cdot \frac{2}{3} - 2 = \frac{5}{6} > \frac{2}{3}$, the only possible cases for (q_1, \dots, q_5) are $(2, 2, 2, 2, 2)$ and $(2, 2, 2, 2, 3)$.

$$(1-B-1) \quad D = K_X + \frac{1}{2}(P_1 + P_2 + \dots + P_5); \quad (2, 2, 5; 10).$$

$$(1-B-2) \quad D = K_X + \frac{1}{2}(P_1 + P_2 + P_3 + P_4) + \frac{2}{3}P_5; \quad (2, 2, 3; 8).$$

Henceforce we assume $r = 4$ and express D by (q_1, q_2, q_3, q_4) and **we always assume** $q_1 \leq q_2 \leq q_3 \leq q_4$. In this case, $a = 2$ and $3 \leq b \leq c$. Hence $\deg D \leq \frac{1}{2}$. Since $4 \cdot \frac{2}{3} - 2 > \frac{1}{2}$, $q_1 = 2$ and $q_4 \geq 3$.

Let s be the number of $q_i > 2$ ($1 \leq s \leq 3$). Then since $\deg[3D] = -6 + 8 - s$, $\dim R_3 = 0, 1, 2$ when $s = 1, 2, 3$, respectively.

If $s = 3$, $\dim R_2 + \dim R_3 = 3$ and we must have $(a, b, c; h) = (2, 3, 3; 9)$.

$$(1-B-3) \quad D = K_X + \frac{1}{2}P_1 + \frac{2}{3}(P_2 + P_3 + P_4); \quad (2, 3, 3; 9).$$

If $s = 2$, $a = 2, b = 3$ and $c \geq 4$. Hence $\deg D = \frac{1}{6} + \frac{1}{c} \leq \frac{5}{12}$. Also, since $-2 + (\frac{1}{2} + \frac{1}{2} + \frac{2}{3} + \frac{3}{4}) = \frac{5}{12}$, we have 2 types.

$$(1-B-4) \quad D = K_X + \frac{1}{2}(P_1 + P_2) + \frac{2}{3}(P_3 + P_4); \quad (2, 3, 6; 12).$$

$$(1-B-5) \quad D = K_X + \frac{1}{2}(P_1 + P_2) + \frac{2}{3}P_3 + \frac{3}{4}P_4; \quad (2, 3, 4; 10).$$

Now we treat the case $(2, 2, 2, q)$, $q \geq 3$. In this case, $R_3 = 0$ and $\dim R_4 = 1$ or 2 according to $q = 3$ or $q \geq 4$. In the latter case, $\dim R_5 = 0$ or 1 according to $q = 4$ or $q \geq 5$. Hence, if $q \geq 5$, we have already 3 generators of R .

$$(1-B-6) \quad D = K_X + \frac{1}{2}(P_1 + P_2 + P_3) + \frac{4}{5}P_4; \quad (2, 4, 5; 12).$$

$$(1-B-7) \quad D = K_X + \frac{1}{2}(P_1 + P_2 + P_3) + \frac{3}{4}P_4; \quad (2, 4, 7; 14).$$

$$(1-B-8) \quad D = K_X + \frac{1}{2}(P_1 + P_2 + P_3) + \frac{2}{3}P_4; \quad (2, 6, 9; 18).$$

We have 8 types in this case.

Case 1 - C. The case $g = 0$ and $r = 3$.

We have to determine (q_1, q_2, q_3) . In this case, $R_1 = R_2 = 0$ and $\dim R_3 = 1$ or 0 according to $q_1 = 2$ or $q_1 \geq 3$.

Case 1. $q_1 \geq 3$.

In this case, $a = 3$ and $4 \leq b \leq c$. Hence $\deg D \leq \frac{1}{4}$. Hence either $q_1 = 3$ or $q_1 = q_2 = q_3 = 4$.

$$(1-C-1) \quad D = K_X + \frac{3}{4}(P_1 + P_2 + P_3); \quad (3, 4, 4; 12).$$

Henceforce we assume $q_1 = 3$.

$R_4 \neq 0$ if and only if $q_2 \geq 4$. In this case, $a = 3, b = 4$ and $c \geq 5$. Hence $\deg D \leq \frac{13}{60} = \frac{2}{3} + \frac{3}{4} + \frac{4}{5} - 2$. Hence we have only 2 possibilities;

$$(1-C-2) \quad D = K_X + \frac{2}{3}P_1 + \frac{3}{4}P_2 + \frac{4}{5}P_3; \quad (3, 4, 5; 13).$$

$$(1-C-3) \quad D = K_X + \frac{2}{3}P_1 + \frac{3}{4}(P_2 + P_3); \quad (3, 4, 8; 16).$$

Next, assume $q_1 = q_2 = 3$. Hence $\deg D = \frac{q_3 - 1}{q_3} - \frac{2}{3}$. On the other hand, since $R_4 = 0$, $a = 3$, $b \geq 5$ and $c \geq 6$ and $\deg D \leq \frac{1}{6}$. This implies $q_3 \leq 6$.

$$(1-C-4) \quad D = K_X + \frac{2}{3}(P_1 + P_2) + \frac{5}{6}P_3; \quad (3, 5, 6; 15).$$

$$(1-C-5) \quad D = K_X + \frac{2}{3}(P_1 + P_2) + \frac{4}{5}P_3; \quad (3, 5, 9; 18).$$

$$(1-C-6) \quad D = K_X + \frac{2}{3}(P_1 + P_2) + \frac{3}{4}P_3; \quad (3, 8, 12; 24).$$

This completes the case $q_1 = 3$.

Case 2. $q_1 = 2$.

In this case, $a \geq 4$ and $R_4 \neq 0$ if and only if $q_2 \geq 4$.

First, we consider the case $q_1 = 2$ and $q_2 = 3$ ($q_3 \geq 7$).

In this case, $\deg[4D] = -1 = \deg[5D] = \deg[7D]$, $\deg[6D] = 0$. Hence $a = 6$ and $b \geq 8$. Hence $\deg D \leq \frac{1}{18} = \frac{8}{9} - \frac{5}{6}$. This shows that $7 \leq q_3 \leq 9$ and actually these cases gives the hypersurfaces.

$$(1-C-7) \quad D = K_X + \frac{1}{2}P_1 + \frac{2}{3}P_2 + \frac{6}{7}P_3; \quad (6, 14, 21; 42).$$

$$(1-C-8) \quad D = K_X + \frac{1}{2}P_1 + \frac{2}{3}P_2 + \frac{7}{8}P_3; \quad (6, 8, 15; 30).$$

$$(1-C-9) \quad D = K_X + \frac{1}{2}P_1 + \frac{2}{3}P_2 + \frac{8}{9}P_3; \quad (6, 8, 9; 24).$$

Next, we consider the case $q_1 = 2$ and $q_2 \geq 4$.

In this case, $\deg[4D] = 0$ and $a = 4, b \geq 5, c \geq 6$. Hence $\deg D \leq \frac{2}{15} = (\frac{1}{2} + \frac{4}{5} + \frac{5}{6}) - 2$. Hence $q_2 \leq 5$ and if $q_2 = 5$, the possibility is the following 2 cases.

$$(1-C-10) \quad D = K_X + \frac{1}{2}P_1 + \frac{4}{5}P_2 + \frac{5}{6}P_3; \quad (4, 5, 6; 16).$$

$$(1-C-11) \quad D = K_X + \frac{1}{2}P_1 + \frac{4}{5}(P_2 + P_3); \quad (4, 5, 10; 20).$$

The remaining case is $q_1 = 2, q_2 = 4$ ($q_3 \geq 5$).

Since $\dim R_4 = 1$ and $R_5 = 0$ and hence $a = 4, b \geq 6, c \geq 7$ and $\deg D = \frac{q_3 - 1}{q_3} - \frac{3}{4} \leq \frac{3}{28}$. Hence $5 \leq q_3 \leq 7$ and actually these cases give hypersurfaces.

This finishes the classification !

$$(1-C-12) \quad D = K_X + \frac{1}{2}P_1 + \frac{3}{4}P_2 + \frac{4}{5}P_3; \quad (4, 10, 15; 30).$$

$$(1-C-13) \quad D = K_X + \frac{1}{2}P_1 + \frac{3}{4}P_2 + \frac{5}{6}P_3; \quad (4, 6, 11; 22).$$

$$(1-C-14) \quad D = K_X + \frac{1}{2}P_1 + \frac{4}{3}P_2 + \frac{6}{7}P_3; \quad (4, 6, 7; 18).$$

The case $\alpha = 2$

We assume that $R = R(X, D) \cong k[u, v, w]/(f)$ with

$$\deg(u, v, w; f) = (a, b, c; h); \quad h = a + b + c + 2.$$

We always assume $(a, b, c) = 1$ and $a \leq b \leq c$. Since R is Gorenstein with $a(R) = 2$, $2D$ is linearly equivalent to $K_X + \text{frac}(D)$. Hence we may assume that

$$D = E + \sum_{i=1}^r \frac{q_i - 1}{2q_i} P_i,$$

where $2E \sim K_X$ and every q_i is odd.

We divide the cases according to (A) $a = b = 1$, (B) $a = 1, b \geq 2$, and (C) $a > 1$.

Case 2 - A. When $a = b = 1$, we have the following 4 types.

$$(2-A-1) \quad g = 6, D = E \text{ with } 2E \sim K_X; \quad (1, 1, 1; 5).$$

$$(2-A-2) \quad g = 4, D = E \text{ with } 2E \sim K_X; \quad (1, 1, 2; 6).$$

$$(2-A-3) \quad g = 3, D = E + \frac{1}{3}P \text{ with } 2E \sim K_X; \quad (1, 1, 3; 7).$$

(2-A-4) $g = 3, D = E$ with $2E \sim K_X$; $(1, 1, 4; 8)$.

Case 2 - B. $a = 1$ and $b \geq 2$

If $b = 2$, we have $g = \dim R_2 = 2$ and we have the following types.

(2-B-1) $g = 2, D = E + \frac{1}{3}P$ with $2E \sim K_X$; $(1, 2, 3; 8)$.

(2-B-2) $g = 2, D = E$ with $2E \sim K_X$; $(1, 2, 5; 10)$.

If $b \geq 3$, then $g = 1$. Since $2E \sim 0$ and $R_1 \neq 0$, we have $E = 0$ and $\deg[3D] = r$. Hence $r \leq 3$ and $\deg D \leq \frac{9}{1 \cdot 3 \cdot 3} = 1$. If $r \geq 2$, then $a = 1$ and $b = 3$. We have the following cases.

(2-B-3) $g = 1, D = \frac{1}{3}(P_1 + P_2 + P_3)$; $(1, 3, 3; 9)$.

(2-B-4) $g = 1, D = \frac{1}{3}P_1 + \frac{2}{5}P_2$; $(1, 3, 5; 11)$.

(2-B-5) $g = 1, D = \frac{1}{3}(P_1 + P_2)$; $(1, 3, 6; 12)$.

If $r = 1$, then $b \geq 5$ and $\deg D \leq \frac{15}{1 \cdot 5 \cdot 7} = \frac{3}{7}$. We have the following cases.

(2-B-6) $g = 1, D = \frac{3}{7}P$; $(1, 5, 7; 15)$.

(2-B-7) $g = 1, D = \frac{2}{5}P$; $(1, 5, 8; 16)$.

(2-B-8) $g = 1, D = \frac{1}{3}P$; $(1, 6, 9; 18)$.

This finishes the case $a = 1, b \geq 2$.

Case 2 - C. $a \geq 2$.

This is equivalent to say that $R_1 = H^0(X, \mathcal{O}_X(D)) = 0$. If this is the case, we have

$$\deg D \leq \frac{9}{2 \cdot 2 \cdot 3} = \frac{3}{4} < 1.$$

Since $\deg D \geq g - 1$, $g = 0$ or 1 in this case.

First, assume $g = 1$. Then $2E \sim 0$ and $\deg E = 0$. Since $\deg[3D] \geq 1$, $R_3 \neq 0$. Hence $a = 2$ and $b = 3$ in this case. We have the following cases.

(2-C-1) $g = 1, D = E + \frac{1}{3}P$ with $E \neq 0, 2E \sim 0$; $(2, 3, 7; 14)$.

(2-C-2) $g = 1, D = E + \frac{2}{5}P$ with $E \neq 0, 2E \sim 0$; $(2, 3, 5; 12)$.

Next, assume $g = 0$. Then since $2E \sim K_X$, $\deg E = -1$ and $\deg[3D] = -3 + r \geq 0$, which implies $a = 3$. Since $\dim R_3 \leq 2$, $r = 3$ or 4 .

If $\dim R_3 = 2$ and $r = 4$, we have only one case.

$$(2-C-3) \quad g = 0, D = E + \frac{1}{3}(P_1 + \dots + P_4) \text{ with } \deg E = -1; \quad (3, 3, 4; 12).$$

If $r = 3$, then $a = 3, b \geq 5$ and $\deg D \leq \frac{15}{3 \cdot 5 \cdot 5} = \frac{1}{5}$. We have the following cases.

$$(2-C-4) \quad g = 0, D = E + \frac{2}{5}(P_1 + P_2 + P_3) \text{ with } \deg E = -1; \quad (3, 5, 5; 15).$$

$$(2-C-5) \quad g = 0, D = E + \frac{1}{3}P_1 + \frac{2}{5}P_2 + \frac{3}{7}P_3 \text{ with } \deg E = -1; \quad (3, 5, 7; 17).$$

$$(2-C-6) \quad g = 0, D = E + \frac{1}{3}P_1 + \frac{2}{5}(P_2 + P_3) \text{ with } \deg E = -1; \quad (3, 5, 10; 20).$$

$$(2-C-7) \quad g = 0, D = E + \frac{1}{3}(P_1 + P_2) + \frac{4}{9}P_3 \text{ with } \deg E = -1; \quad (3, 7, 9; 21).$$

$$(2-C-8) \quad g = 0, D = E + \frac{1}{3}(P_1 + P_2) + \frac{3}{7}P_3 \text{ with } \deg E = -1; \quad (3, 7, 12; 24).$$

$$(2-C-9) \quad g = 0, D = E + \frac{1}{3}(P_1 + P_2) + \frac{2}{5}P_3 \text{ with } \deg E = -1; \quad (3, 10, 15; 30).$$

This finishes the case $a(R) = 2$.

The case $\alpha = 3$

We assume that $R = R(X, D) \cong k[u, v, w]/(f)$ with

$$\deg(u, v, w; f) = (a, b, c; h); \quad h = a + b + c + 3.$$

Since R is Gorenstein with $a(R) = 3$, $3D$ is linearly equivalent to $K_X + \text{frac}(D)$. By 1.1 (1), we may assume that

$$(3.1.1) \quad D = E + \sum_{i=1}^r \frac{q_i - 1}{3q_i} P_i + \sum_{i=1}^s \frac{2q_i - 1}{3q_i} Q_i.$$

By 1.1 (1), we have

$$(3.1.2) \quad 3E + \sum_{i=1}^s Q_i \sim K_X; \quad \deg E = \frac{2g - 2 - s}{3}.$$

Since $\frac{q_i - 1}{3q_i} \geq \frac{1}{4}$ if $q_i \equiv 1(\text{mod } 3)$, $q_i > 1$ and $\frac{2q_i - 1}{3q_i} \geq \frac{1}{2}$ if $q_i \equiv 2(\text{mod } 3)$, we have

$$(3.1.3) \quad \deg D \geq \deg E + \frac{r}{4} + \frac{s}{2} = \frac{2g - 2}{3} + \frac{r}{4} + \frac{s}{6}.$$

We divide the cases according to (A) $a = b = 1$, (B) $a = 1, b \geq 2$, (C) $a \geq 2$.

Case 3 - A. $a = b = 1$.

In this case, the type of R is the form $(1, 1, c; c + 5)$. By Lemma 1.2, $c \leq 5$ and $c \neq 3$. So we have for cases $c = 1, 2, 4, 5$. We can calculate the genus g by $g = \dim R_3$.

(3-A-1) $g = 10$, $D = E$ with $\deg E = 6$, $3E \sim K_X$; $(1, 1, 1; 4)$. X is not hyperelliptic.

(3-A-2) $g = 6$, $D = E + \frac{1}{2}Q$ with $\deg E = 3$, $3E + Q \sim K_X$; $(1, 1, 2; 7)$.

Since $g = 6$ and $\deg D = \frac{7}{2} = \frac{10}{3} + \frac{1}{6}$, this is the only possible equation for D .

(3-A-3) $g = 4$, $D = E + \frac{1}{4}P$ with $\deg E = 2$, $3E \sim K_X$; $(1, 1, 4; 9)$.

(3-A-4) $g = 4$, $D = E$ with $\deg E = 2$, $3E \sim K_X$; $(1, 1, 5; 10)$.

Case 3 - B. $a = 1, b \geq 2$.

In this case, since $R_1 = H^0(X, O_X(E)) \neq 0$, we may assume $E \geq 0$. Since $g = \dim R_3$, we have $g \leq 3$. Also, $g \geq 2$ if and only if $b \leq 3$ and $g = 1$ otherwise.

If $g = 3$, then $c \leq 3$ and we have the following cases.

(3-B-1) $g = 3$, $D = \frac{1}{2} \sum_{i=1}^4 Q_i$ with $\sum_{i=1}^4 Q_i \sim K_X$; $(1, 2, 2; 8)$.

(3-B-2) $g = 3$, $D = E + \frac{1}{2}Q$ with $\deg E = 1$, $3E + Q \sim K_X$; $(1, 2, 3; 9)$.

If $g = 2$, then $b \leq 3$ and $c \geq 4$. Also, by (3.1.2), $E = 0$, $s = 2$ and $Q_1 + Q_2 \sim K_X$. Hence $\dim R_2 = h^0(K_X) = 2$. Thus we have $b = 2$.

Since $E = 0$, $\deg[4D] = r + 2s \geq 4$ and $c = 4$ if and only if $r > 0$.

(3-B-3) $g = 2$, $D = \frac{1}{4}P + \frac{1}{2}(Q_1 + Q_2)$ with $Q_1 + Q_2 \sim K_X$; $(1, 2, 4; 10)$.

Now, since $E = 0$ and $s = 2$, we have $\deg D \geq 1$. On the other hand, since $a = 1, b = 2$, $\deg D = \frac{c+6}{2c}$ and $\deg D \geq 1$ if and only if $c \leq 6$. Hence we have the following cases.

(3-B-4) $g = 2$, $D = \frac{3}{5}Q_1 + \frac{1}{2}Q_2$; with $Q_1 + Q_2 \sim K_X$; $(1, 2, 5; 11)$.

(3-B-5) $g = 2$, $D = \frac{1}{2}(Q_1 + Q_2)$ with $Q_1 + Q_2 \sim K_X$; $(1, 2, 6; 12)$.

In the remaining cases, $a = 1$ and $b \geq 4$. Then we have $g = 1$ and by (3.1.2) we have $E = 0$ and $s = 0$. Also, since $\deg[4D] = r$, $b = 4$ if and only if $r \geq 2$. If this is the case, we have the following 3 cases.

(3-B-6) $g = 1$, $D = \frac{1}{4}(P_1 + P_2 + P_3)$; $(1, 4, 4; 12)$.

(3-B-7) $g = 1$, $D = \frac{1}{4}P_1 + \frac{2}{7}P_2$; $(1, 4, 7; 15)$.

$$(3-B-8) \quad g = 1, D = \frac{1}{4}(P_1 + P_2); \quad (1, 4, 8; 16).$$

If $g = 1, s = 0$ and $r = 1, b \geq 7$ and $c \geq 10$. Hence $\deg D \leq \frac{21}{70} = \frac{3}{10}$. We have the following 3 types.

$$(3-B-9) \quad g = 1, D = \frac{1}{4}P; \quad (1, 8, 12; 24).$$

$$(3-B-10) \quad g = 1, D = \frac{2}{7}P; \quad (1, 7, 11; 22).$$

$$(3-B-11) \quad g = 1, D = \frac{3}{10}P; \quad (1, 7, 10; 21).$$

Case 3 - C. $a \geq 2$.

In this case, $\deg D \leq \frac{10}{2 \cdot 2 \cdot 3} < 1$.

Since $g = \dim R_3, g \leq 2$. But since the hypersurfaces of type $(3, 3, c; c+9)$ cannot be normal, $g \leq 1$.

If $g = 1$, by (3.1.2), either $\deg E = 0 = s$ or $\deg E = -1$ and $s = 3$. In the first case, since $3E \sim 0, R_1 = 0, E \not\sim 0$ and $R_2 = 0$. Hence $\deg D \leq \frac{15}{3 \cdot 4 \cdot 5} = \frac{1}{4}$ and we have the following case.

$$(3-C-1) \quad g = 1, D = E + \frac{1}{4}P, 3E \sim 0, E \not\sim 0; \quad (3, 4, 5; 15).$$

In the latter case, since $\deg[2D] \geq 1$, we have $a = 2, b = 3$. Since $\deg D \leq \frac{12}{2 \cdot 3 \cdot 4} = \frac{1}{2}$, the following case is the only possibility.

$$(3-C-2) \quad g = 1, D = E + \frac{1}{2}(P_1 + P_2 + P_3), 3E + P_1 + P_2 + P_3 \sim 0; \quad (2, 3, 4; 12).$$

Now, until the end of the case $\alpha = 2$, we assume $g = 0$. By (3.1.2), we have $\deg E = -\frac{s+2}{3}$ and $\deg[2D] = s + 2\deg E = \frac{s-4}{3}$.

If $s = 7$, then $a = b = 2$ and $\deg D \geq \frac{1}{2}$, which implies $c \leq 7$ and we have the following 2 cases.

$$(3-C-3) \quad g = 0, D = E + \frac{1}{2}(P_1 + \dots + P_6) + \frac{3}{5}P_7, \deg E = -3; \quad (2, 2, 5; 12).$$

$$(3-C-4) \quad g = 0, D = E + \frac{1}{2}(P_1 + \dots + P_6 + P_7), \deg E = -3; \quad (2, 2, 7; 14).$$

If $s = 4$ and $\deg E = -2$, we have $a = 2$ and since $\deg[4D] = r \leq 1, b = 4$ if and only if $r = 1$, otherwise, $b \geq 5$. If $r = 1$, then $\deg D \geq \frac{1}{4}$ and $a = 2, b = 4$ and $c \leq 9$ and we have the following 3 cases.

$$(3-C-5) \quad g = 0, D = E + \frac{1}{4}P + \frac{3}{5}Q_1 + \frac{1}{2}(Q_2 + Q_3 + Q_4), \deg E = -2; \quad (2, 4, 5; 14).$$

$$(3-C-6) \quad g = 0, D = E + \frac{2}{7}P + \frac{1}{2}(Q_1 + \dots + Q_4), \deg E = -2; \quad (2, 4, 7; 16).$$

$$(3-C-7) \quad g = 0, D = E + \frac{1}{4}P + \frac{1}{2}(Q_1 + \dots + Q_4), \deg E = -2; \quad (2, 4, 9; 18).$$

If $s = 4$ and $r = 0$, then $b \geq 5$ and we have $\frac{1}{10} \leq \deg D \leq \frac{15}{2 \cdot 5 \cdot 5}$, we have the following 6 cases.

$$(3-C-8) \quad g = 0, D = E + \frac{1}{2}Q_1 + \frac{3}{5}(Q_2 + Q_3 + Q_4), \deg E = -2; \quad (2, 5, 5; 15).$$

$$(3-C-9) \quad g = 0, D = E + \frac{5}{8}Q_1 + \frac{3}{5}Q_2 + \frac{1}{2}(Q_3 + Q_4), \deg E = -2; \quad (2, 5, 8; 18).$$

$$(3-C-10) \quad g = 0, D = E + \frac{3}{5}(Q_1 + Q_2) + \frac{1}{2}(Q_3 + Q_4), \deg E = -2; \quad (2, 5, 10; 20).$$

$$(3-C-11) \quad g = 0, D = E + \frac{7}{11}Q_1 + \frac{1}{2}(Q_2 + Q_3 + Q_4), \deg E = -2; \quad (2, 8, 11; 24).$$

$$(3-C-12) \quad g = 0, D = E + \frac{5}{8}Q_1 + \frac{1}{2}(Q_2 + Q_3 + Q_4), \deg E = -2; \quad (2, 8, 13; 26).$$

$$(3-C-13) \quad g = 0, D = E + \frac{3}{5}Q_1 + \frac{1}{2}(Q_2 + Q_3 + Q_4), \deg E = -2; \quad (2, 10, 15; 30).$$

If $s = 1$, since $\deg(K_X + \text{frac}(D)) > 0$, we must have $r \geq 2$. On the other hand, since $R_2 = R_3 = 0$ and $\deg[4D] = r - 2 \leq 1$, $r \leq 3$. Hence $a \geq 4$ and $\deg D \leq \frac{16}{4 \cdot 4 \cdot 5} = \frac{1}{5}$. If $r \geq 3$, then $\deg D \geq \frac{1}{4}$. Hence $r = 2$, $a = 4$, $b \geq 5$. We have the following 7 cases.

$$(3-C-14) \quad g = 0, D = E + \frac{3}{5}Q + \frac{2}{7}P_1 + \frac{1}{4}P_2, \deg E = -1; \quad (4, 5, 7; 19).$$

$$(3-C-15) \quad g = 0, D = E + \frac{5}{8}Q + \frac{1}{4}(P_1 + P_2), \deg E = -1; \quad (4, 5, 8; 20).$$

$$(3-C-16) \quad g = 0, D = E + \frac{3}{5}Q + \frac{1}{4}(P_1 + P_2), \deg E = -1; \quad (4, 5, 12; 24).$$

$$(3-C-17) \quad g = 0, D = E + \frac{1}{2}Q_1 + \frac{2}{7}P_1 + \frac{3}{10}P_2, \deg E = -1; \quad (4, 7, 10; 24).$$

$$(3-C-18) \quad g = 0, D = E + \frac{1}{2}Q_1 + \frac{2}{7}(P_1 + P_2), \deg E = -1; \quad (4, 7, 14; 28).$$

$$(3-C-19) \quad g = 0, D = E + \frac{1}{2}Q_1 + \frac{1}{4}P_1 + \frac{3}{10}P_2, \deg E = -1; \quad (4, 10, 17; 34).$$

$$(3-C-20) \quad g = 0, D = E + \frac{1}{2}Q_1 + \frac{1}{4}P_1 + \frac{2}{7}P_2, \deg E = -1; \quad (4, 14, 21; 42).$$

This finishes the case $a(R) = 3$.

The case $\alpha = 4$

We assume that $R = R(X, D) \cong k[u, v, w]/(f)$ with

$$\deg(u, v, w; f) = (a, b, c; h); \quad h = a + b + c + 4.$$

Since $4D \sim K_X + \text{frac}(D)$, we may assume that

$$(4.1.1) \quad D = E + \sum_{i=1}^r \frac{q_i - 1}{4q_i} P_i + \sum_{i=1}^s \frac{3q_i - 1}{4q_i} Q_i,$$

where E is an integral divisor on X .

Since $4D \sim K_X + \text{frac}(D)$, we have

$$(4.1.2) \quad 4E + 2 \sum_{i=1}^s Q_i \sim K_X; \quad \deg E = \frac{g - 1 - s}{2}.$$

Since $\frac{q_i - 1}{4q_i} \geq \frac{1}{5}$ if $q_i \equiv 1 \pmod{4}$, $q_i > 1$ and $\frac{3q_i - 1}{4q_i} \geq \frac{2}{3}$ if $q_i \equiv 3 \pmod{4}$, we have

$$(4.1.3) \quad \deg D \geq \deg E + \frac{r}{5} + \frac{2s}{3} = \frac{g - 1}{2} + \frac{r}{5} + \frac{s}{6}.$$

We divide the cases according to (A) $a = b = 1$, (B) $a = 1, b \geq 2$, (C) $a \geq 2$.

Case A. $a = b = 1$.

In this case, the type of R is the form $(1, 1, c; c + 6)$. By 1.2, $c \leq 6$ and we have the following cases.

(4-A-1) $g = 15$, $D = E$ with $\deg E = 6$, $3E \sim K_X$; $(1, 1, 1; 4)$. X is not hyperelliptic.

(4-A-2) $g = 9$, $D = E$ with $\deg E = 4$, $4E \sim K_X$; $(1, 1, 2; 8)$.

(4-A-3) $g = 7$, $D = E$ with $\deg D = 3$, $3E \sim K_X$; $(1, 1, 3; 9)$.

(4-A-4) $g = 5$, $D = E + \frac{1}{5}$ with $\deg E = 2$, $4E \sim K_X$; $(1, 1, 4; 10)$.

(4-A-5) $g = 5$, hyperelliptic, $D = E$ with $\deg D = 2$, $4D \sim K_X$; $(1, 1, 6; 12)$.

Case B. $a = 1, b \geq 2$.

In this case, since $R_1 = H^0(X, O_X(E)) \neq 0$, we may assume $E \geq 0$. Since $g = \dim R_4$, $g \geq 2$ if and only if $b \leq 4$ and $g = 1$ otherwise. Checking the cases of type $(1, 2, c; c + 7)$ and $(1, 3, c; c + 8)$, we have the following types.

(4-B-1) $g = 4$, $D = E + \frac{2}{3}Q$ with $4E + 2Q \sim K_X$; $(1, 2, 3; 10)$.

(4-B-2) $g = 3$, $D = E + \frac{1}{5}Q$ with $4E \sim K_X$; $(1, 2, 5; 12)$.

(4-B-3) $g = 3$, hyperelliptic; $D = E$ with $4E \sim K_X$; $(1, 2, 7; 14)$.

(4-B-4) $g = 3$, $D = E$ with $4E \sim K_X$; $(1, 3, 4; 12)$.

$$(4-B-5) \quad g = 2, D = \frac{1}{5}P + \frac{2}{3}Q \text{ with } 2Q \sim K_X; \quad (1, 3, 5; 13).$$

$$(4-B-6) \quad g = 2, D = \frac{5}{7}Q \text{ with } 2Q \sim K_X; \quad (1, 3, 7; 15).$$

$$(4-B-7) \quad g = 2, D = \frac{2}{3}Q \text{ with } 2Q \sim K_X; \quad (1, 3, 8; 16).$$

These finishes the case $a = 1$ and $2 \leq b \leq 3$. We can also check that the type $(1, 4, c; c + 9)$ can not give a normal ring. Hence we may assume $b \geq 5$ and hence $g = \dim R_4 = 1$. Also, we have $\deg D \leq \frac{15}{5 \cdot 5} = \frac{3}{5}$. Hence $E = 0$ and $s = 0$. Also, $b = 5$ if and only if $r \geq 2$. Since $\dim R_5 \leq 3$, we have also $r \leq 3$ and if $r = 3$, then $\deg D \geq \frac{3}{5}$.

$$(4-B-8) \quad g = 1, D = \frac{1}{5}(P_1 + P_2 + P_3); \quad (1, 5, 5; 15).$$

If $r = 2$, $b = 5$ and $c \geq 9$. Hence $\deg D \leq \frac{16}{45} = \frac{1}{5} + \frac{2}{9}$.

$$(4-B-9) \quad g = 1, D = \frac{1}{5}P_1 + \frac{2}{9}P_2; \quad (1, 5, 9; 19).$$

$$(4-B-10) \quad g = 1, D = \frac{1}{5}(P_1 + P_2); \quad (1, 5, 10; 20).$$

If $r = 1$, $b \geq 9$ and $c \geq 13$. Hence $\deg D \leq \frac{27}{9 \cdot 13} = \frac{3}{13}$. Thus we have only the following possibilities.

$$(4-B-11) \quad g = 1, D = \frac{1}{5}P; \quad (1, 10, 15; 30).$$

$$(4-B-12) \quad g = 1, D = \frac{2}{9}P; \quad (1, 9, 14; 28).$$

$$(4-B-13) \quad g = 1, D = \frac{3}{13}P; \quad (1, 9, 13; 27).$$

This finishes the case $R_1 \neq 0$.

Case C. $a \geq 2$.

Since $\deg D \leq \frac{11}{2 \cdot 2 \cdot 3} < 1$, we have $g \leq 2$ by (4.1.3). But if $\dim R_4 \geq 2$, 2 among a, b, c should be even and R will not be normal. Hence $g \leq 1$.

If $g = 1$, then s is even and $\deg E = -\frac{s}{2}$ by (4.1.2). Also by (4.1.3), $s \leq 4$ and we have always $\deg[2D] = 0$. Hence $\deg D \leq \frac{12}{2 \cdot 3 \cdot 3} = \frac{2}{3}$. If $s = 4$, $\deg D \geq \frac{2}{3}$ and we have equality.

$$(4-C-1) \quad g = 1, D = E + \frac{2}{3}(P_1 + P_2 + P_3 + P_4), \deg E = -2, -2E \sim P_1 + P_2 + P_3 + P_4; \quad (2, 3, 3; 12).$$

If $g = 1, \deg E = -1$ and $s = 2, \deg D \geq \frac{1}{3}$ by (5.3.1). On the other hand, if $R_2 = 0$, then $\deg D \leq \frac{16}{3 \cdot 4 \cdot 5}$ and we have a contradiction. Thus we must have $\dim R_2 = 1$, that is, $2E + P_1 + P_2 \sim 0$. If $r \geq 1$, then $\deg D \geq \frac{2}{3} + \frac{1}{5}$ and on the other hand, $\deg D \leq \frac{16}{3 \cdot 4 \cdot 5}$ and we have a contradiction. If $r = 0$, the only possibility is the following;

$$(4-C-2) \quad g = 1, D = E + \frac{2}{3}P_1 + \frac{5}{7}P_2, \deg E = -1, -2E \sim P_1 + P_2; \quad (2, 3, 7; 16).$$

If $g = 1, s = 0$ and $\deg E = 0$, then $\deg D \leq \frac{16}{2 \cdot 5 \cdot 5} < \frac{1}{3}$ and we must have $r = 1$. Since we must have $\dim R_4 = 1$, the following case is the only possible one.

$$(4-C-3) \quad g = 1, D = E + \frac{2}{9}P, E \neq 0, 2E \sim 0; \quad (2, 5, 9; 20).$$

Now, we assume $g = 0$. Since $R_4 = 0, a \geq 3$ and in (4.1.1) we have $\deg E = \frac{-s-1}{2}$ by (4.1.2). Since $\deg[3D] = \frac{s-3}{2} \leq 1$, we get $s \leq 5$. If $s = 5$, we have the following.

$$(4-C-4) \quad g = 0, D = E + \frac{2}{3}(Q_1 + \dots Q_5), \deg E = -3, \quad (3, 3, 5; 15).$$

Next, we assume $s = 3, \deg E = -2$. In this case, $a = 3, b \geq 5, \dim R_5 = r$ and $\dim R_6 = r + 1$. Hence if $r > 0, r = 1, (a, b, c; h) = (3, 5, 6; 18)$ and $\deg D = \frac{1}{5}$.

$$(4-C-5) \quad g = 0, D = E + \frac{2}{3}(Q_1 + Q_2 + Q_3) + \frac{1}{5}P, \deg E = -2, \quad (3, 5, 6; 18).$$

If $r = 0, \deg[7D] = -2 + t$, where t is the number of q'_i 's with $q_i \geq 7$. We have the following cases;

$$(4-C-6) \quad g = 0, D = E + \frac{5}{7}(Q_1 + Q_2 + Q_3), \deg E = -2, \quad (3, 7, 7; 21).$$

$$(4-C-7) \quad g = 0, D = E + \frac{2}{3}Q_1 + \frac{5}{7}(Q_2 + Q_3), \deg E = -2, \quad (3, 7, 14; 28).$$

$$(4-C-8) \quad g = 0, D = E + \frac{2}{3}Q_1 + \frac{5}{7}Q_2 + \frac{8}{11}Q_3, \deg E = -2, \quad (3, 7, 11; 25).$$

$$(4-C-9) \quad g = 0, D = E + \frac{2}{3}(Q_1 + Q_2) + \frac{11}{15}Q_3, \deg E = -2, \quad (3, 11, 15; 33).$$

$$(4-C-10) \quad g = 0, D = E + \frac{2}{3}(Q_1 + Q_2) + \frac{8}{11}Q_3, \deg E = -2, \quad (3, 11, 18; 36).$$

$$(4-C-11) \quad g = 0, D = E + \frac{2}{3}(Q_1 + Q_2) + \frac{5}{7}Q_3, \deg E = -2, \quad (3, 14, 21; 42).$$

Then we treat the case $s = 1, \deg E = -1$. Since $s + r \geq 3, r \geq 2$ and we have $a = 5, b = 6$ and the only possible case is;

$$(4\text{-C-12}) \quad g = 0, D = E + \frac{2}{3}Q + \frac{1}{5}(P_1 + P_2), \deg E = -1, \quad (5, 6, 15; 30).$$

This finishes the classification of the case with $a(R) = 4$.

The case $\alpha = 5$

We assume that $R = R(X, D) \cong k[u, v, w]/(f)$ with

$$\deg(u, v, w; f) = (a, b, c; h); \quad h = a + b + c + 5.$$

Since $5D$ is linearly equivalent to $K_X + \text{frac}(D)$, we may assume that

$$(5.1.1) \quad \begin{aligned} D = & E + \sum_{i=1}^r \sum_{q_i \equiv 1 \pmod{5}} \frac{q_i - 1}{5q_i} P_i + \sum_{i=1}^s \sum_{q_i \equiv 3 \pmod{5}} \frac{2q_i - 1}{5q_i} P'_i \\ & + \sum_{k=1}^t \sum_{q_k \equiv 2 \pmod{5}} \frac{3q_k - 1}{5q_k} Q_k + \sum_{l=1}^u \sum_{q_l \equiv 4 \pmod{5}} \frac{4q_l - 1}{5q_l} Q'_l, \end{aligned}$$

where E is an integral divisor on X .

Since $5D \sim K_X + \text{frac}(D)$, we have

$$(5.1.2) \quad \begin{aligned} 5E + \sum_{j=1}^s P'_j + 2 \sum_{k=1}^t Q_k + 3 \sum_{l=1}^u Q'_l & \sim K_X; \\ \deg E = \frac{2g - 2 - s - 2t - 3u}{5} & \geq \frac{2g - 2}{5} + \frac{r}{6} + \frac{2s}{15} + \frac{t}{10} + \frac{3u}{20}. \end{aligned}$$

We divide the cases according to (A) $a = b = 1$, (B) $a = 1, b \geq 2$, (C) $a \geq 2$.

Case A. $a = b = 1$.

In this case, the type of R is the form $(1, 1, c; c + 7)$. By 1.2, $c \leq 7$ and we have the following cases.

$$(5\text{-A-1}) \quad g = 21, D = E \text{ with } \deg E = 8, 5E \sim K_X; \quad (1, 1, 1; 8).$$

$$(5\text{-A-2}) \quad g = 12, D = E + \frac{1}{2}P \text{ with } \deg E = 4, 5E + 2P \sim K_X; \quad (1, 1, 2; 9).$$

$$(5\text{-A-3}) \quad g = 9, D = E + \frac{1}{3}P \text{ with } \deg E = 3, 5E + P \sim K_X; \quad (1, 1, 3; 10).$$

$$(5\text{-A-4}) \quad g = 6, D = E + \frac{1}{6}P \text{ with } \deg E = 2, 5E \sim K_X; \quad (1, 1, 6; 13).$$

$$(5\text{-A-5}) \quad g = 6, D = E \text{ with } \deg E = 2, 5E \sim K_X; \quad (1, 1, 7; 14).$$

Case B. $a = 1, b \geq 2$.

First, we list the cases $a = 1$ and $b = 2, 3$.

$$(5\text{-B-1}) \quad g = 6, D = \frac{1}{2}(P_1 + \dots + P_5) \text{ with } 2(P_1 + \dots + P_5) \sim K_X; \quad (1, 2, 2; 10).$$

$$(5\text{-B-2}) \quad g = 5, D = E + \frac{1}{2}P \text{ with } \deg E = 4, 5E + 2P \sim K_X; \quad (1, 2, 3; 11).$$

$$(5-B-3) \quad g = 4, D = \frac{1}{2}(P_1 + P_2 + P_3) \text{ with } 2(P_1 + P_2 + P_3) \sim K_X; \quad (1, 2, 4; 12).$$

$$(5-B-4) \quad g = 3, D = \frac{1}{2}P_1 + \frac{1}{3}(P_2 + P_3) \text{ with } 2P_1 + P_2 + P_3 \sim K_X; \quad (1, 2, 6; 14).$$

$$(5-B-5) \quad g = 3, D = \frac{1}{2}P_1 + \frac{4}{7}P_2 \text{ with } 2(P_1 + P_2) \sim K_X; \quad (1, 2, 7; 15).$$

$$(5-B-6) \quad g = 3, D = \frac{1}{2}(P_1 + P_2) \text{ with } 2(P_1 + P_2) \sim K_X; \quad (1, 2, 8; 16).$$

$$(5-B-7) \quad g = 3, D = \frac{1}{3}(P_1 + \dots + P_4) \text{ with } (P_1 + \dots + P_4) \sim K_X; \quad (1, 3, 3; 12).$$

$$(5-B-8) \quad g = 3, D = \frac{1}{3}P_1 + \frac{3}{4}P_2 \text{ with } P_1 + 3P_2 \sim K_X; \quad (1, 3, 4; 13).$$

$$(5-B-9) \quad g = 2, D = \frac{1}{3}(P_1 + P_2) + \frac{1}{6}P_3 \text{ with } P_1 + P_2 \sim K_X; \quad (1, 3, 6; 15).$$

$$(5-B-10) \quad g = 2, D = \frac{1}{3}(P_1 + P_2) \text{ with } P_1 + P_2 \sim K_X; \quad (1, 3, 9; 18).$$

If $a = 1$ and $b \geq 4$, then $\deg D < 1$ and $g \leq 2$. If $g = 2 = \dim R_5$, then $b \leq 5$.

$$(5-B-11) \quad g = 2, D = \frac{1}{2}P_1 + \frac{1}{6}P_2 \text{ with } 2P_1 \sim K_X; \quad (1, 4, 6; 16).$$

$$(5-B-12) \quad g = 2, D = \frac{1}{2}P \text{ with } 2P \sim K_X; \quad (1, 4, 10; 20).$$

In the case $a = 1$ and $b \geq 6$, then $D \geq 0$, $\deg D \leq \frac{1}{2}$ and $g = 1$.

$$(5-B-13) \quad g = 1, D = \frac{1}{6}(P_1 + P_2 + P_3); \quad (1, 6, 6; 18).$$

$$(5-B-14) \quad g = 1, D = \frac{1}{6}(P_1 + P_2); \quad (1, 6, 12; 24).$$

If $a = 1$ and $b \geq 7$, then $\deg D < \frac{1}{3}$. Hence only possibility is of the form $D = \frac{q-1}{5q}$ with $q \equiv 1 \pmod{5}$.

$$(5-B-15) \quad g = 1, D = \frac{2}{11}; \quad (1, 11, 17; 34).$$

$$(5-B-16) \quad g = 1, D = \frac{3}{16}P; \quad (1, 11, 16; 33).$$

$$(5-B-17) \quad g = 1, D = \frac{1}{6}P; \quad (1, 12, 18; 36).$$

This finishes the case $a = 1$.

Case C. $a \geq 2$.

In this case, $\deg D \leq \frac{12}{2 \cdot 2 \cdot 3} = 1$. Since $g = \dim R_5$, $g \leq 2$ and $g = 2$ only in the following cases.

$$(5-C-1) \quad g = 2, D = E + \frac{1}{2}(P_1 + P_2), \text{ with } E \neq 0, 2E + P_1 + P_2 \sim K_X; \quad (2, 2, 3; 12).$$

$$(5-C-2) \quad g = 2, D = \frac{1}{6}P; \quad (2, 3, 5; 15).$$

If $g = 1$, $a = 2$ then we have the following cases.

$$(5-C-3) \quad g = 1, D = E + \frac{1}{2}P_1 + \frac{1}{2}P_2 + \frac{1}{3}P_3 + E, \text{ with } \deg E = -1, 2E + P_1 + P_2 \sim E + P_3 \sim 0; \quad (2, 3, 7; 17).$$

$$(5-C-4) \quad g = 1, D = \frac{3}{8}P + E \text{ with } 2E \sim 0, 5E + P \sim 0; \quad (2, 3, 8; 18).$$

$$(5-C-5) \quad g = 1, D = \frac{1}{3}P + E \text{ with } 2E \sim 0, 5E + P \sim 0; \quad (2, 3, 10; 20).$$

If $g = 1$ and $a \geq 3$, we have $5E \sim 0$ since $\dim R_5 = 1$ and by (6.1.2), $s = t = u = 0$. Since $E \neq 0$, $R_n = 0$ for $1 \leq n \leq 4$ and we have $a = 5, b \geq 6$. Hence $\deg D \leq \frac{22}{5 \cdot 6 \cdot 6} < \frac{1}{6}$. On the other hand, since $r > 0$, we should have $\deg E \geq \frac{1}{6}$. Hence there is no case with $g = 1, a \geq 3$.

Now we treat the case $g = 0$. We divide our discussion into the following cases

$$(i) R_2 \neq 0, \quad (ii) R_2 = 0 \text{ and } R_3 \neq 0, \quad (iii) R_2 = R_3 = 0.$$

(i) If $\dim R_2 = 2$, we have the following cases.

$$(5-C-6) \quad g = 0, D = E + \frac{1}{2}(P_1 + \dots + P_8) + \frac{4}{7}P_9 \text{ with } \deg E = -4; \quad (2, 2, 7; 16).$$

$$(5-C-7) \quad g = 0, D = E + \frac{1}{2}(P_1 + \dots + P_9) \text{ with } \deg E = -4; \quad (2, 2, 9; 18).$$

If $a = 2, b = 4$, we have the following cases.

$$(5-C-8) \quad g = 0, D = E + \frac{1}{2}(P_1 + \dots + P_4) + \frac{4}{7}P_5 + \frac{3}{4}P_6 \text{ with } \deg E = -3; \quad (2, 4, 7; 18).$$

$$(5-C-9) \quad g = 0, D = E + \frac{1}{2}(P_1 + \dots + P_5) + \frac{7}{9}P_6 \text{ with } \deg E = -3; \quad (2, 4, 9; 20).$$

$$(5-C-10) \quad g = 0, D = E + \frac{1}{2}(P_1 + \dots + P_5) + \frac{3}{4}P_6 \text{ with } \deg E = -3; \quad (2, 4, 11; 22).$$

If $a = 2$ and $b \geq 6$, from $\deg[2D] = \deg[4D] = 0$ and $\deg[3D] = -2$ we have $\deg E = -2, t = 4, s = u = 0$. $b = 6$ if and only if $r > 0$.

$$(5-C-11) \quad g = 0, D = E + \frac{1}{2}(P_1 + \dots + P_3) + \frac{4}{7}P_4 + \frac{1}{6}P_5 \text{ with } \deg E = -2; \quad (2, 6, 7; 20).$$

$$(5-C-12) \quad g = 0, D = E + \frac{1}{2}(P_1 + \dots + P_4) + \frac{2}{11}P_5 \text{ with } \deg E = -2; \quad (2, 6, 11; 24).$$

$$(5-C-13) \quad g = 0, D = E + \frac{1}{2}(P_1 + \dots + P_4) + \frac{1}{6}P_5 \text{ with } \deg E = -2; \quad (2, 6, 13; 26).$$

If $a = 2$ and $b \geq 7$, then we have $\deg E = -2, t = 4, r = s = u = 0$.

$$(5-C-14) \quad g = 0, D = E + \frac{1}{2}P_1 + \frac{4}{7}(P_2 + P_3 + P_4) \text{ with } \deg E = -2; \quad (2, 7, 7; 21).$$

$$(5-C-15) \quad g = 0, D = E + \frac{1}{2}(P_1 + P_2) + \frac{4}{7}(P_3 + P_4) \text{ with } \deg E = -2; \quad (2, 7, 14; 28).$$

$$(5-C-16) \quad g = 0, D = E + \frac{1}{2}(P_1 + P_2) + \frac{4}{7}(P_3 + P_4) \text{ with } \deg E = -2; \quad (2, 7, 14; 28).$$

$$(5-C-17) \quad g = 0, D = E + \frac{1}{2}(P_1 + P_2 + P_3) + \frac{10}{17}P_4 \text{ with } \deg E = -2; \quad (2, 12, 17; 36).$$

$$(5-C-18) \quad g = 0, D = E + \frac{1}{2}(P_1 + P_2 + P_3) + \frac{7}{12}P_4 \text{ with } \deg E = -2; \quad (2, 12, 19; 38).$$

$$(5-C-19) \quad g = 0, D = E + \frac{1}{2}(P_1 + P_2 + P_3) + \frac{4}{7}P_4 \text{ with } \deg E = -2; \quad (2, 14, 21; 42).$$

This finishes the case $g = 0$ and $a = 2$.

(ii) The case $g = 0$ and $a = 3$.

If $a = 3$ and $b = 3$ or 4 , we have the following cases.

$$(5-C-20) \quad g = 0, D = E + \frac{1}{3}(P_1 + \dots + P_5) + \frac{3}{4}P_6 \text{ with } \deg E = -2; \quad (3, 3, 4; 15).$$

$$(5-C-21) \quad g = 0, D = E + \frac{3}{4}(P_1 + \dots + P_4) + \frac{1}{3}P_5 \text{ with } \deg E = -3; \quad (3, 4, 4; 16).$$

$$(5-C-22) \quad g = 0, D = E + \frac{1}{3}P_1 + \frac{3}{4}(P_2 + P_3) + \frac{3}{8}P_4 \text{ with } \deg E = -2; \quad (3, 4, 8; 20).$$

$$(5-C-23) \quad g = 0, D = E + \frac{1}{3}(P_1 + P_2) + \frac{3}{4}P_3 + \frac{7}{9}P_4 \text{ with } \deg E = -2; \quad (3, 4, 9; 21).$$

$$(5-C-24) \quad g = 0, D = E + \frac{1}{3}(P_1 + P_2) + \frac{3}{4}(P_3 + P_4) \text{ with } \deg E = -2; \quad (3, 4, 12; 24).$$

If $a = 3$ and $b \geq 6$, then since $\deg[D] = \deg[4D] = -1$ by Lemma 0.3 and $\deg[3D] = 0$, we have $\deg E = -1, s = 3, t = u = 0$. Also, in this case $r > 0$ if and only if $b = 6$. In the latter case, since $\deg D \leq \frac{21}{3 \cdot 6 \cdot 7} = \frac{1}{6}$, $D = E + \frac{1}{3}(P_1 + P_2 + P_3) + \frac{1}{6}P_4$ with $\deg E = -1$.

$$(5-C-25) \quad g = 0, D = E + \frac{1}{3}(P_1 + P_2 + P_3) + \frac{1}{6}P_4 \text{ with } \deg E = -1; \quad (3, 6, 7; 21).$$

If $r = 0$, we have the following cases.

$$(5-C-26) \quad g = 0, D = E + \frac{3}{8}(P_1 + P_2 + P_3) \text{ with } \deg E = -1; \quad (3, 8, 8; 24).$$

$$(5-C-27) \quad g = 0, D = E + \frac{1}{3}P_1 + \frac{3}{8}P_2 + \frac{5}{13}P_3 \text{ with } \deg E = -1; \quad (3, 8, 13; 29).$$

$$(5-C-28) \quad g = 0, D = E + \frac{1}{3}P_1 + \frac{3}{8}(P_2 + P_3) \text{ with } \deg E = -1; \quad (3, 8, 16; 32).$$

$$(5-C-29) \quad g = 0, D = E + \frac{1}{3}(P_1 + P_2) + \frac{7}{18}P_3 \text{ with } \deg E = -1; \quad (3, 13, 18; 39).$$

$$(5-C-30) \quad g = 0, D = E + \frac{1}{3}(P_1 + P_2) + \frac{5}{13}P_3 \text{ with } \deg E = -1; \quad (3, 13, 21; 42).$$

$$(5-C-31) \quad g = 0, D = E + \frac{1}{3}(P_1 + P_2) + \frac{3}{8}P_3 \text{ with } \deg E = -1; \quad (3, 16, 24; 48).$$

This finishes the case $g = 0$ and $a = 3$.

(iii) $g = 0, R_2 = R_3 = 0$.

First we treat the case $R_4 \neq 0$. It is easy to see that hypersurfaces of type $(4, 4, c; c + 13)$ can not be normal. So, if $a = 4$, then $n \geq 6$ and we have $\deg E = -2, s + u = 2, t + u = 3$. If $a = 4$ and $b = 6$, we have the following types.

$$(5-C-32) \quad g = 0, D = E + \frac{1}{2}(P_1 + P_2) + \frac{1}{3}P_3 + \frac{7}{9}P_4 \text{ with } \deg E = -2; \quad (4, 6, 9; 24).$$

$$(5-C-33) \quad g = 0, D = E + \frac{1}{2}(P_1 + P_2) + \frac{1}{3}P_3 + \frac{3}{4}P_4 \text{ with } \deg E = -2; \quad (4, 6, 15; 30).$$

If $a = 4$ and $b > 6$, then $\deg[6D] < 0$. Since $5D \sim K_X + \text{frac}(D)$, this is only possible if the support of $\text{frac}(D)$ consists of 3 points. This implies $r = s = 0, t = 1, u = 2$.

$$(5-C-34) \quad g = 0, D = E + \frac{1}{2}P_1 + \frac{7}{9}P_2 + \frac{11}{14}P_3 \text{ with } \deg E = -2; \quad (4, 9, 14; 32).$$

$$(5-C-35) \quad g = 0, D = E + \frac{1}{2}P_1 + \frac{7}{9}(P_2 + P_3) \text{ with } \deg E = -2; \quad (4, 9, 18; 36).$$

$$(5-C-36) \quad g = 0, D = E + \frac{1}{2}P_1 + \frac{3}{4}P_2 + \frac{7}{9}P_3 \text{ with } \deg E = -2; \quad (4, 18, 27; 54).$$

$$(5-C-37) \quad g = 0, D = E + \frac{1}{2}P_1 + \frac{3}{4}P_2 + \frac{11}{14}P_3 \text{ with } \deg E = -2; \quad (4, 14, 23; 46).$$

$$(5-C-38) \quad g = 0, D = E + \frac{1}{2}P_1 + \frac{3}{4}P_2 + \frac{15}{19}P_3 \text{ with } \deg E = -2; \quad (4, 14, 19; 42).$$

This finishes the case $a = 4$. If $a > 4$, then $a \geq 6$ and since $\deg[nD] = -1$ by Lemma 1.4, we have $\deg E = -1$. Since $6D = (K_X + \text{frac}(D)) + D$, $\deg[6D] = -3 + r + s + t + u \geq 0$. Hence $a = 6$ and it is easy to see type $(6, 6, c; c + 17)$ does not occur. Hence we have either $r = 2, s = t = 0, u = 1$ or $r = s = t = 1, u = 0$. In the former case, $\deg D \geq 2\frac{1}{6} + \frac{3}{4} - 1 = \frac{1}{12}$. On the other hand, since $b \geq 7$ and $c \geq 8$, $\deg D \leq \frac{26}{6 \cdot 7 \cdot 8} < \frac{1}{12}$. So, this case does not occur and we have $r = s = t = 1, u = 0$. We have the following cases.

$$(5-C-39) \quad g = 0, D = E + \frac{1}{3}P_1 + \frac{1}{6}P_2 + \frac{4}{7}P_3 \text{ with } \deg E = -1; \quad (6, 7, 9; 27).$$

$$(5-C-40) \quad g = 0, D = E + \frac{1}{2}P_1 + \frac{3}{8}P_2 + \frac{2}{11}P_3 \text{ with } \deg E = -1; \quad (6, 8, 11; 30).$$

$$(5-C-41) \quad g = 0, D = E + \frac{1}{2}P_1 + \frac{1}{6}P_2 + \frac{5}{13}P_3 \text{ with } \deg E = -1; \quad (6, 8, 13; 32).$$

$$(5-C-42) \quad g = 0, D = E + \frac{1}{2}P_1 + \frac{1}{3}P_2 + \frac{2}{11}P_3 \text{ with } \deg E = -1; \quad (6, 22, 33; 66).$$

$$(5-C-43) \quad g = 0, D = E + \frac{1}{2}P_1 + \frac{1}{3}P_2 + \frac{3}{16}P_3 \text{ with } \deg E = -1; \quad (6, 16, 27; 54).$$

$$(5-C-44) \quad g = 0, D = E + \frac{1}{2}P_1 + \frac{1}{3}P_2 + \frac{4}{21}P_3 \text{ with } \deg E = -1; \quad (6, 16, 21; 48).$$

This finishes the case $a(R) = 5$.

The case $\alpha = 6$

Since $6D$ is linearly equivalent to $K_X + \text{frac}(D)$, we may assume that

$$(6.1.1) \quad D = E + \sum_{i=1}^r \frac{q_i - 1}{6q_i} P_i + \sum_{i=1}^s \frac{5q_i - 1}{6q_i} Q_i,$$

where E is an integral divisor on X .

Since $6D \sim K_X + \text{frac}(D)$, we have

$$(6.1.2) \quad 6E + 4 \sum_{i=1}^s Q_i \sim K_X; \quad \deg E = \frac{g - 1 - 2s}{3}.$$

Since $\frac{q_i - 1}{6q_i} \geq \frac{1}{7}$ if $q_i \equiv 1 \pmod{6}$, $q_i > 1$ and $\frac{5q_i - 1}{6q_i} \geq \frac{4}{5}$ if $q_i \equiv 5 \pmod{6}$, we have

$$(6.1.3) \quad \deg D \geq \deg E + \frac{r}{7} + \frac{4s}{5} = \frac{g - 1}{3} + \frac{r}{7} + \frac{2s}{15}.$$

We divide the cases according to (A) $a = b = 1$, (B) $a = 1, b \geq 2$, (C) $a \geq 2$.

Case A. $a = b = 1$.

In this case, we have the following cases. We can calculate the genus g by $g = \dim R_6$.

(6-A-1) $g = 28$, $D = E$ with $\deg E = 9$, $6E \sim K_X$; $(1, 1, 1; 9)$. X is not hyperelliptic.

(6-A-2) $g = 16$, $D = E$ with $\deg E = 5$, $6E \sim K_X$; $(1, 1, 2; 10)$. X is not hyperelliptic.

(6-A-3) $g = 10$, $D = E$ with $\deg E = 3$, $6E \sim K_X$; $(1, 1, 4; 12)$.

(6-A-4) $g = 7$, $D = E + \frac{1}{7}$ with $\deg E = 2$, $6E \sim K_X$; $(1, 1, 7; 15)$.

(6-A-5) $g = 7$, $D = E$ with $\deg E = 2$, $6E \sim K_X$; $(1, 1, 8; 16)$.

Case B. $a = 1, b \geq 2$.

First we list the cases with $a = 1, b = 2, 3, 4$.

(6-B-1) $g = 7, D = E$ with $\deg E = 2, 6E \sim K_X; (1, 2, 3; 12)$.

(6-B-2) $g = 4, D = E + \frac{1}{7}$ with $\deg E = 1, 6E \sim K_X; (1, 2, 7; 16)$.

(6-B-3) $g = 4, X$ is hyperelliptic, $D = E$ with $\deg E = 1, 6E \sim K_X; (1, 2, 9; 18)$.

(6-B-4) $g = 4, D = E$ with $\deg E = 1, 6E \sim K_X; (1, 3, 5; 15)$.

(6-B-5) $g = 3, D = \frac{4}{5}P$ with $4P \sim K_X; (1, 4, 5; 16)$.

If $b \geq 5$, then $\deg D \leq \frac{3}{5}$. Hence $E = 0$ and $s = 0$. This implies $\deg[nD] = 0$ for $n \leq 6$. Thus we have $g = 1$ and $b \geq 7, \deg D \leq \frac{3}{7}$.

(6-B-6) $g = 1, D = \frac{1}{7}(P_1 + P_2 + P_3); (1, 7, 7; 21)$.

(6-B-7) $g = 1, D = \frac{1}{7}(P_1 + P_2 + P_3); (1, 7, 7; 21)$.

(6-B-6) $g = 1, D = \frac{1}{7}(P_1 + P_2); (1, 7, 14; 28)$.

(6-B-7) $g = 1, D = \frac{1}{7}P_1 + \frac{2}{13}P_2; (1, 7, 13; 27)$.

(6-B-8) $g = 1, D = \frac{3}{19}P; (1, 13, 19; 39)$.

(6-B-9) $g = 1, D = \frac{2}{13}P; (1, 13, 20; 40)$.

(6-B-10) $g = 1, D = \frac{1}{7}P; (1, 14, 21; 42)$.

This finishes the case $a = 1, b \geq 2$.

Case C. $a \geq 2$.

We can easily see that $(a, b) = (2, 2), (2, 3), (2, 4), (3, 3)$ does not occur. By a similar computation, we can assert $\deg D < \frac{1}{3}$ and we know that $g \leq 1$.

If $g = 1$, then $6E + 4\sum_{i=1}^s Q_j \sim 0$. If $\deg E = -2$, then $s = 3$ and $\deg D \geq \frac{2}{5}$, which contradicts our previous computation. Hence, if $g = 1$, then $\deg E = 0$ and $s = 0$. This implies that $a = 2, 3$ or 6 and $\deg[nD] = 0$ for $1 \leq n \leq 6$. We have the following cases.

(6-C-1) $g = 1, D = E + \frac{1}{7}P$ with $E \neq 0, 2E \sim 0; (2, 7, 15; 30)$.

(6-C-2) $g = 1, D = E + \frac{2}{13}P$ with $E \neq 0, 2E \sim 0; (2, 7, 13; 28)$.

(6-C-3) $g = 1, D = E + \frac{1}{7}P$ with $E \neq 0, 3E \sim 0; (3, 7, 8; 24)$.

This finishes the case $g = 1$. If $g = 0$, we have

$$6E + 4 \sum_{i=1}^s Q_j \sim K_X, \quad \deg E = \frac{-1 - 2s}{3}.$$

If $\deg E = -1, s = 1$, then $R_n = 0$ for $n \leq 6$ and this implies $\deg D \leq \frac{28}{7 \cdot 7 \cdot 8} = \frac{1}{14}$. On the other hand, since $r + s \geq 3$, $\deg D \geq -1 + \frac{4}{5} + \frac{2}{7} = \frac{6}{35}$, a contradiction !

Hence we have $s = 4$ and $\deg E = -3$. Then $\deg[4D] = 0$ and $\deg[5D] = 1$. Hence we are restricted to the following type.

(6-C-4) $g = 0, D = E + \frac{4}{5}(P_1 + \dots + P_4)$ with $\deg E = -3; \quad (4, 5, 5; 20)$.

This finishes the case $a(R) = 6$.

Now, we list the table of numbers of cases with given $a(R) \leq 6$. In the table, br denotes the number of branches of D , which is equal to the number of branches of the graph of the minimal resolution of $\text{Spec}(R)$. In other word, $\text{br} = \deg[\text{frac}(D)]$.

$a(R)$	1	2	3	4	5	6
$g = 0, \text{br} = 3$	14	6	7	7	11	0
$g = 0, \text{br} \geq 4$	8	1	10	2	22	1
$g = 1$	6	8	8	7	8	8
$g = 2$	2	2	3	3	6	0
$g = 3$	1	2	2	3	5	1
$g \geq 4$	0	2	4	6	8	9
Total	31	21	34	28	58	19

REFERENCES

- [Dem] Demazure, M.: Anneaux gradués normaux ; in Seminarire Demazure -Giraud -Teissier, 1979. Ecole Polytechnique In:Lê Dũng Tráng(ed.) Introduction a la théorie des singularités II; Méthodes algébriques et géométriques (Travaux En Cours vol.**37**, pp.35-68) Paris:Hermann 1988
- [GW] Goto, S., Watanabe, K.-i.: On graded rings I. J. Math. Soc. Japan. **30** (1978) 179-213.
- [Dol] Dolgačev, I.V., Automorphic forms, and quasihomogeneous singularities, Funcional Anal. Appl. **9** (1975), 149-151.
- [OW] Orlik, P. and Wagreich, P., Isolated singularities of algebraic surfaces with \mathbb{C}^* -action, Annals. Math., **93** (1971), 205-228.
- [P1] Pinkham, H.: Singularités exceptionnelles, la dualité etrange d'Arnold et les surfaces K3, C.R.Acad.Sc.Paris, **284** (A) (1977), 615-618.
- [P2] Pinkham, H.: Normal surface singularities with \mathbb{C}^* -action Math. Ann. **227**, (1977) 183-193

- [S0] Saito, K.: Quasihomogene isolierte Singularitäten von Hyperflächen, *Inventiones Math.* **14** (1971), 123-142.
- [S1] Saito, K.: Regular system of weights and associated singularities, In: Suwa, T., Wagreich, Ph. (ed.) *Complex analytic singularities.* (Adv. Studies in pure Math. **8**, 479-526) Tokyo Amsterdam: Kinokuniya-North-Holland 1986
- [S2] Saito, K.: On the existence of exponents prime to the Coxeter number, *J. of Algebra*, **114** (1986), 333-356.
- [To] M. Tomari, Multiplicity of filtered rings and simple K3 singularities of multiplicity two, *Publ. Res. Inst. Math. Sci.* **38** (2002), 693–724.
- [Wag] P. Wagreich; Algebras of automorphic forms with few generators. *Trans. Amer. Math. Soc.* **262** (1980), no. 2, 367–389.
- [W] Watanabe, K.-i.: Some remarks concerning Demazure’s construction of normal graded rings, *Nagoya Math. J.* **83**, (1981) 203-211

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